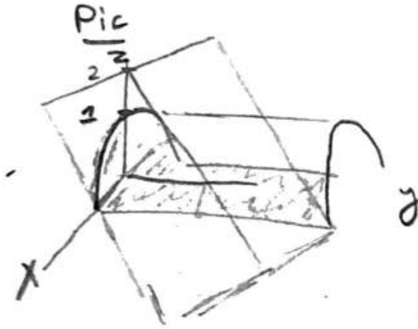
Recall Divergence Thm

If R & ∂R are nice and $\vec{F}$ is a vf on $\mathbb{R}^3$ with cts partial derivatives, then

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV.$$

Ex. Compute the Flux of $\vec{F} = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$ across S the surface of region $z = 1 - x^2$, $z = 0$, $y = 0$, $y + z = 2$

Pic



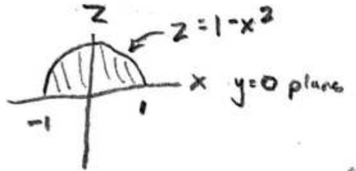
Sol: $\operatorname{div}(\vec{F}) = \frac{d}{dx}(xy) + \frac{d}{dy}(y^2 + e^{xz^2}) + \frac{d}{dz}(\sin(xy))$

$= y + 2y + 0 = 3y$

$y = 2 - z$

$R = \{(x,y,z) : -1 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 2 - z$

region



$$\iiint_R \operatorname{div}(\vec{F})\, dV$$

$$= 3 \int_{x=-1}^{1} \int_{z=0}^{1-x^2} \int_{y=0}^{2-z} y \, dy\, dz\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} \int_{0}^{1-x^2} y^2 \Big|_0^{2-z} dz\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} -\frac{1}{3}(2-z)^3 \Big|_0^{1-x^2} dx$$

$$= \frac{3}{2} \cdot -\frac{1}{3} \int_{-1}^{1} (1+x^2)^3 - 8 \, dx$$

$$= -\frac{1}{2} \int_{-1}^{1} (1 + 3x^2 + 3x^4 + x^6 - 8)\, dx$$

odd function
$f(-x) = f(x)$

$$= -\frac{1}{2}\left(-7x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7\right)\Big|_{-1}^{1}$$

$$= -\frac{1}{2} \cdot 2\left(-7 + 1 + \frac{3}{5} + \frac{1}{7}\right)$$

Ex. Compute Flux of $\vec{F} = \langle xye^z, xy^2z^3, -ye^z \rangle$ across the box bounded by coordinate planes { $x=3$ $y=2$ $z=1$

Sol: $R = [0,3] \times [0,2] \times [0,1]$

$$\operatorname{div}(\vec{F}) = \frac{d}{dx}(xye^z) + \frac{d}{dy}(xy^2z^3) + \frac{d}{dz}(-ye^z)$$

$$= ye^z + 2xyz^3 - ye^z$$

$$= 2xyz^3$$

$$F = 2\int_0^3\int_0^2\int_0^1 xyz^3\,dz\,dy\,dx$$

$$= \cancel{2}\cdot\frac{1}{\cancel{4}_2}\cdot 2\cdot\frac{9}{\cancel{2}} = \frac{9}{2}$$

Ex. Compute Flux of $\vec{F} = \langle 2x^3+y^3, y^3+z^3, 3y^2z \rangle$ across surface of region bounded by the paraboloid $z = 1-x^2-y^2$ and plane $z = -3$

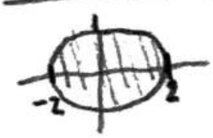
Pic

Sol. $\operatorname{div}(\vec{F}) = 6x^2 + 3y^2 + 3y^2$

$= 6(x^2+y^2) = 6r$

$z = 1-x^2-y^2$

Shadow $z=-3$



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$R(r,\theta,z)$: $0<r<2$, $0<\theta<2\pi$, $-3<z<1-r^2$

$$\int_0^{2\pi}\int_0^2\int_{-3}^{1-r^2} 6r\cdot r\,dz\,dr\,d\theta \quad 6r^2(1-r^2--3) = 6r^2(4-r^2)$$

$24r^2-6r^4$

$$\int_0^{2\pi}\int_0^2 24r^2-6r^4\,dr\,d\theta = 6\int_0^{2\pi}\left(16-\frac{32}{3}\right)d\theta$$

$$= 96-64\int_0^{2\pi} d\theta$$

$$= 32(2\pi) = 64\pi$$

Ex. Compute flux of $\vec{F} = \langle z, y, zx \rangle$ across the surface of tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

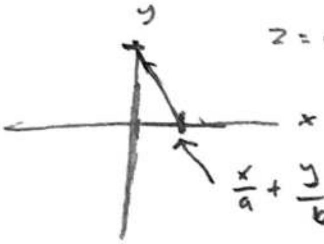
Sol.

$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \vec{x} = \langle x, y, z \rangle$

$\vec{n} \cdot \vec{x} - \vec{n} \cdot \vec{p} = 0$

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$\vec{n} = \langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \rangle$



$z = 0$

$R = \{(x,y,z) : 0 \le x \le a,\ 0 \le y \le b(1-\frac{x}{a}),\ 0 \le z \le c(1-\frac{x}{a}-\frac{y}{b})\}$

$z = c(1 - \frac{x}{a} - \frac{y}{b})$

$\frac{x}{a} + \frac{y}{b} = 1 \qquad y = b(1 - \frac{x}{a})$

$\frac{x}{a} = 1 \quad x = a \qquad \operatorname{div}(\vec{F}) = \langle 0, +1, +x \rangle = 1 + x$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\, dz\, dy\, dx$$

$$= \int_0^a (1+x) \cdot \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= c \int_0^a (1+x) \left[y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b(1-\frac{x}{a})} = c \int_0^a (1+x) \left[b(1-\tfrac{x}{a}) \right] \left(1 - \frac{x}{a} - \frac{b}{2b}(1 - \frac{x}{a})\right)$$

$$= bc\frac{1}{2} \int_0^a (1+x)(1-\tfrac{x}{a}) \cdot (1 - \tfrac{x}{a})\, dx$$

$$= \frac{1}{2} bc \int_0^a \left(1 + (1-\tfrac{2}{a})x + (\tfrac{1}{a^2} - \tfrac{2}{a})x^2 + \tfrac{1}{a^2} x^3\right) dx$$

$$= \frac{1}{2} bc \left[x + \frac{1}{2}(1-\tfrac{2}{a})x^2 + \frac{1}{3}(\tfrac{1}{a^2} - \tfrac{2}{a})x^3 + \frac{1}{4a^2} x^4 \right]_0^a$$

$$= \frac{1}{2} bc \left(a + \frac{1}{2}(1-\tfrac{2}{a})a^2 + \frac{1}{3}(a^{-2} - \tfrac{2}{a})a^3 + \frac{a^4}{4a^2} \right)$$

$$= \frac{1}{2} abc \left(1 + \frac{1}{2}(a-2) + \frac{1}{3}(1-2a) + \frac{1}{4}a \right)$$

$$= \frac{1}{2} abc \left(\frac{1}{3} + a(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}) \right) = \frac{1}{6} abc \left(1 + (\tfrac{6}{4} - \tfrac{8}{4} + \tfrac{1}{4})a \right)$$

$$= \frac{1}{6} abc (1 - \tfrac{1}{4}a).$$

$\blacksquare$